Numerical Analysis of Heat Conduction Problems on Irregular Domains by means of a Meshless Method

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Introduction

Standard numerical methods for PDEs (e.g., FEM and FV) require domain meshing, which is a time consuming process, especially with complex shaped/moving/deformable domains. **Meshless methods require only some distribution of points over the domain to solve the problem**.

Target of the work

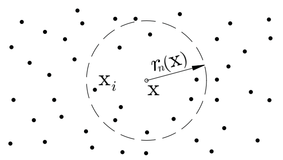
In this work we focus our attention on the numerical solution of a 2D Poisson equation (**steady state heat conduction problem**):

$$\nabla^2 \phi = q \qquad \text{in } \Omega$$

with Dirichlet boundary conditions (fixed temperature) along $\partial\Omega$: $\phi = \bar{\phi}$.

Numerical Method

A **Collocation Meshless Method** based on **Radial Basis Functions (RBF)** is employed. Only a single set of points $\mathbf{x}_i$ distributed over the domain has to be generated. A **locally supported RBF interpolation** is used to approximate the unknown field ϕ and its derivatives through the nearest n points :



$$\phi(\mathbf{x}) = \sum_{i=1}^{n} a_i \varphi(\|\mathbf{x} - \mathbf{x}_i\|) + \mathbf{b} \cdot \mathbf{x} + c$$

where the RBFs φ are *Hardy's Multiquadrics* (MQ) functions:

$$\varphi(r) = \sqrt{(\varepsilon r)^2 + 1}$$

ε is the *shape factor* whose choice is crucial because of its strong influence on results; therefore this choice has to be made case by case.

Point Distributions

Two point distribution techniques have been employed in this work:

- Specific geometry dependent approach;
- Quadtree technique.

The first approach, feasible only with simple geometries, aims to generate high quality distributions that will be used as reference, while the quadtree technique is a more general and widely used approach for practical problems. The quality of quadtree-generated points is then improved using a simple refinement process based on mutual repulsion of points; this quadtree approach is **fully automated**.

References

[1] B. Šarler, R. Vertnik: *Meshfree Explicit Local Radial Basis Function Collocation Method for Diffusion Problems*, Computers & Mathematics with Applications, **51**(8), 1269-1282, (2006)

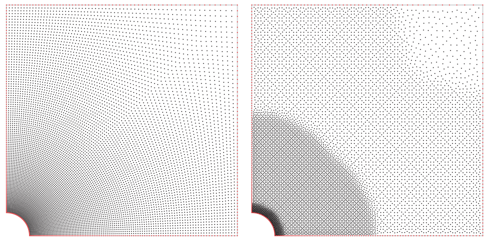
[2] G. Kosec, B. Šarler: *Adaptive Meshfree Method For Thermo-fluid Problems With Phase Change*, Advances in Fluid Mechanics VIII, **69**, 91-101, (2010)

Results

For the first test case the reference solution is:

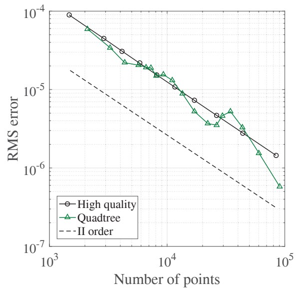
$$\bar{\phi} = \sqrt{r}$$

where r is the radius from the center of the bottom left arc; since $\bar{\phi}$ exhibits an increasing (negative) gradient around this arc, as can be seen in the contour plot below, the point distribution density has been increased in the same way to capture this behaviour, as visible from side pictures. Convergence curves (below) confirm that with $n = 8$ nearest points the accuracy order for both point distributions is 2.



High quality distribution — Quadtree distribution

Solution $\bar{\phi}$

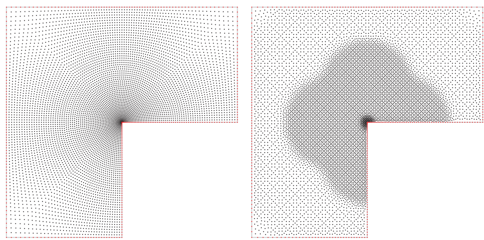


While the convergence curve for the high quality distribution shows an **exact 2nd order accuracy**, the convergence curve for the quadtree approach has a very similar behaviour. Therefore, in this case where the solution has no unbounded gradients, the **quadtree approach is comparable with the high quality distribution approach**, and sometimes it can even perform better.

For the second test case the reference solution is:

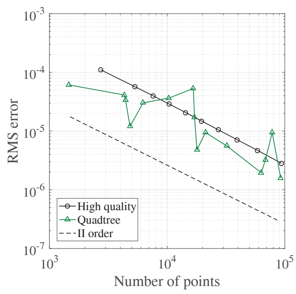
$$\bar{\phi} = r^{\beta} \sin(\beta\theta)\ ,\ \beta = 2/3$$

r and θ are, respectively, the radius and the horizontal angle from the central corner. This solution is representative of a reentrant corner whose internal surface is kept at a constant temperature, as can be seen from the contour plot below. The reentrant corner causes a singolarity (unbounded radial derivatives in $r = 0$), whose resolution is possible increasing the point distribution density around $r = 0$ in an appropriate way.



High quality distribution — Quadtree distribution

Solution $\bar{\phi}$

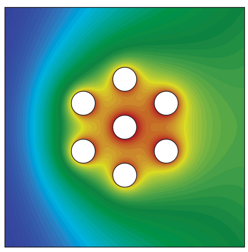


Therefore this is a more severe test than the previous one. From convergence curves aside we can see that the use of the high quality distribution allows the recovery of an **exact 2nd order accuracy** even in the presence of a singularity. On the contrary, the quadtree distribution curve has a segmented behaviour, suggesting that the solution strongly depends on the specific point distribution around the corner (quadtree point distribution density has a discontinuous behaviour).

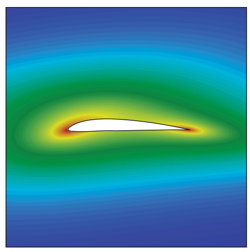
However, even if its behaviour is not ideal, the mean trend is **absolutely comparable with the the high quality one** in terms of absolute values, suggesting that the quadtree approach for the generation of points for meshless methods is feasible for practical problems.

Conclusions and Future work

The simple test cases reported in this work suggest that the refined quadtree generation process employed for the generation of point distributions can be successfully employed with a collocation meshless approach to solve diffusion equations. The **degree of accuracy** obtained in these cases is **comparable (and sometimes better) to the one obtained with high quality point distributions**, generated specifically case by case. Futher investigations will be carried out in order to ensure that this specific meshless method, coupled with the refined quadtree generation process, can be employed to correctly simulate complex problems with practical geometries, as can be seen in the two examples above. The employed approach has a **natural and straightforward extension to the 3D case** since RBF interpolation is independent of dimensionality, while the the point distribution process can be obtained via an octree technique. Future work will also concern the extension to different steady/unsteady advection-diffusion problems such as **Navier-Stokes** equations.



Domain with holes



Airfoil